STRUCTURAL DAMAGE DETECTION USING DECENTRALIZED CONTROLLER DESIGN METHOD

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ABSTRACT

Observer-based fault detection and isolation (FDI) filter design method is a model-based method. By carefully choosing the observer gain, the residual outputs can be projected into different independent subspaces. Each subspace corresponds to different monitored structural element so that the projected residual will be nonzero when the associated structural element is damaged and zero when there is no damage. The key point of detection filter design is how to find an appropriate observer gain. This problem can be interpreted in a geometric language and is found to be equivalent to the problem of finding a decentralized static output feedback gain. But it is a challenging task to find the decentralized controller by either analytical or numerical methods because its solution set is generally non-convex. In this paper, the concept of detection filter and iterative LMI method for decentralized controller design are combined to develop an algorithm to compute the observer gain. It can be used to monitor structural element state: healthy or damaged. The simulation result shows that the developed method can successfully identify structural damages.

Keywords: Structural damage detection, Decentralized control

INTRODUCTION

Existence of structural damage in civil engineering infrastructures, such as high-rise buildings, highway/railway bridges, offshore petroleum foundations, etc., may greatly influence the overall performance of the system or even lead to disastrous consequences. Therefore, detecting structural damages, which are caused by earthquakes, impacts, or explosions immediately after the event or monitoring long-term deterioration due to environmental changes and human uses, is very important for structural maintenance. This leads to the field of research known as structural damage detection, or structural health monitoring, or alternatively, fault detection, isolation and identification.

In the past decades, numerous approaches to the problem of Failure Detection and Isolation (FDI) in dynamic systems have been developed. Among them are two major FDI philosophies: physical redundancy and analytical redundancy. Physical redundancy is achieved simply through hardware replication. Unlike physical redundancy, analytical redundancy, which implies the inherent redundancy contained in the static and dynamic relationship among the system inputs and measured outputs (Frank, 1990), is a model-based method and has gained increasing consideration world-wide recently. Analytical redundancy methods have many advantages over physical redundancy methods, for example the replication of identical hardware components (actuator/sensor) is more expensive, restricted, and sometimes difficult to implement in practice (Dharap et al., 2006; Koh et al., 2005a, 2005b).

There are many FDI methods based on analytical redundancy approaches. Among them, the Beard-Jones detection filter (BJDT) has received increasing consideration world-wide recently. In their pioneering work done in the early seventies, Beard (1971) and Jones (1973) found that with the proper choice of filter feedback gains, the filter residual will have directional characteristics that can be easily associated with different faults. The BJDT filter design method has been successively improved by many people, e.g. Massoumnia (1986), White (1987), Douglas (1996, 1999) and Liberatore (2002). In

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particular, an important geometric interpretation of the BJDT filter has been developed by Massoumnia (1986). Douglas (1996, 1999) extended this geometric interpretation and found that the problem of finding a detection filter gain is equivalent to that of finding a constant static decentralized output feedback controller, and then Youla parameterization was used to obtain those detection filters. In this paper, this equivalent static decentralized output feedback controller design problem is solved by iterative linear matrix inequalities (ILMI) method, which is introduced by Cao et al. (1998).

Decentralized control is widely used in large-scale systems, e.g. electric power networks, socioeconomic systems and large-scale space stations, which are usually geographically distributed. Centralized control of such systems is either uneconomical or unreliable due to long-distance information transfer between local control stations. Decentralized control only uses the locally measurements to compute control inputs, which reduces the risk of data losing and time delay during long-distance information transfer. However, it is a challenging task to find decentralized controllers by either analytical or numerical methods because its solution set is generally non-convex. Recently, linear matrix inequalities (LMIs) approach has been proposed to solve the decentralized stabilization and H_∞ control problem, for example Cao et al. (1998), Scorletti and Duc (2001), Zhai et al. (2001) and reference therein. Cao et al. (1998) studied the static output feedback decentralized stabilization problem, proposed an iterative LMI (ILMI) algorithm to obtain the decentralized feedback gain and extended the idea to static output feedback stabilization with guaranteed H_∞ performance. Scorletti and Duc (2001) modeled the linear time invariant (LTI) system with decentralized controllers as an interconnection of subsystems. Dissipative concept and LMI approach were combined to design the controller for each closed-loop subsystem. Zhai et al. (2001) considered the dynamic decentralized output feedback H_∞ control problem and reduced it to a feasibility problem of a bilinear matrix inequality (BMI) which was solved by using the idea of the homotopy method.

In this paper, the concept of detection filter design (Douglas, 1996, 1999) and the ILMI method for decentralized controller design (Cao et al., 1998) are combined to develop an algorithm for structural damage detection and isolation. The system is assumed to be observable. The state observer is constructed to generate the residual outputs, which contain the structural damage information. By carefully choosing the observer gain, the residual outputs can be projected into different independent subspaces. Each subspace is related to different monitored structural element. The problem of finding the observer gain is converted to that of finding a static decentralized output feedback controller. ILMI approach is used to find the stable observer gain which put the poles of the closed-loop system to the left of some specified negative number to improve FDI system performance. The simulation result shows that the developed method can successfully identify structural damages.

The paper is organized as follows. Section 2 is divided into two parts. The first part explains the iterative LMI method for decentralized controller design problem. The second part describes the observer-based detection filter design method and then applies iterative LMI procedure to find the detection gain for structural damage detection and isolation. Section 3 introduces a ten-story shear type building example to illustrate the applicability of the FDI method presented in this paper. Section 4 concludes this paper.

MATHEMATICAL FORMULATIONS

Decentralized Static Output Feedback Controller Design Using ILMI Approach

This section gives a brief review of decentralized static output feedback controller design based on ILMI approach (Cao et al., 1998). Consider a LTI system with q control channels

$$\begin{aligned} \dot{x} &= Ax + \sum_{i=1}^{q} B_i u_i \\ z &= C_1 x \\ y_i &= C_{2i} x, \quad i = 1, 2, \cdots, q \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, z is the controlled output, u_i and y_i are the control input and the measured output of channel i (i = 1,2,…,q). B_i is an $n \times n_{ui}$ input influence matrix, C_1 is an $n_z \times n$ controlled output influence matrix, C_{2i} is an $n_{yi} \times n$ measured output influence matrix.

The decentralized static output feedback control law is characterized as

$$u_i = F_i y_i, \quad i = 1, 2, \cdots, q \tag{2}$$

where F_i is an $n_{ui} \times n_{yi}$ constant output feedback gain. The controlled input u_i only depends on y_i, which is part of the whole measurements y.

If we stack input u and output y as: $u = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_q^T \end{bmatrix}^T$, $y = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_q^T \end{bmatrix}^T$, then the system (1) and control law (2) can be rewritten as a closed-loop form as

$$\begin{aligned} \dot{x} &= \left(A + BF_D C_2\right) x \\ z &= C_1 x \end{aligned} \tag{3}$$

where: $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_q \end{bmatrix}$, $C_2 = \begin{bmatrix} C_{21}^T & C_{22}^T & \cdots & C_{2q}^T \end{bmatrix}^T$, $F_D = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_q \end{bmatrix}$

Notice that the static output feedback gain F_D is a block diagonal matrix in spite of a full matrix as seen in centralized control. The closed-loop system (3) is stabilizable if there exists a block diagonal matrix F_D such that all eigenvalues of ($A + BF_DC$) are in the left complex plane. However, even for centralized static output feedback control it is very difficult to find a stable controller since the solution set is non-convex (Ghaoui et al., 1997). Obviously, it is more difficult to obtain the decentralized static output feedback controller F_D because of its block diagonal constraint. Cao et al. (1998) presented an iterative LMI approach to solve this problem; they proved the following necessary and sufficient condition for decentralized static output feedback stability.

There exists a decentralized static output feedback controller F_D such that the closed-loop system is stable and all poles of the closed-loop system are put to the left of Re(s) = α / 2 in the complex plane if and only if there exist two symmetric and positive definite matrix P and X of compatible dimensions satisfying the matrix inequality

$$\begin{bmatrix} A^T P + PA - XBB^T P - PBB^T X + XBB^T X - \alpha P & \left(B^T P + F_D C\right)^T \\ \left(B^T P + F_D C\right) & -I \end{bmatrix} < 0 \tag{4}$$

where α is a negative number.

The matrix inequality (4) is non-linear since there exist non-linear terms, such as XBB^TP, PBB^TX and XBB^TX. Thus, the difficulty is that how to find a block diagonal matrix F_D such that non-linear matrix inequality (4) is satisfied with some $P > 0$ and $X > 0$. It can not be solved directly using Matlab LMI toolbox. This is the reason why iterative LMI approach is needed. The iterative algorithm consists of the following steps.

Step 1: Select $Q > 0$, and solve the following Riccati equation:

$$A^T P + PA - PBB^T P + Q = 0 \tag{5}$$

Assume the solution is X and set $i = 1$.

Step 2: Substitute X into the matrix inequality (6) and solve the generalized eigenvalue problem for α_i.

$$\begin{bmatrix} A^T P_i + P_i A - XBB^T P_i - P_i BB^T X + XBB^T X - \alpha P_i & \left(B^T P_i + F_D C\right)^T \\ \left(B^T P_i + F_D C\right) & -I \end{bmatrix} < 0 \tag{6}$$

$$P_i = P_i^T > 0 \tag{7}$$

Step 3: Substitute X and α_i into the matrix inequality (6) and solve the optimization problem for P_{i} and F_{D}: Minimize trace(P_{i}) subjected to the LMI constraints (6) and (7).

Step 4: If α_i or all eigenvalues of $(A + BF_D C)$ are less than some specified negative number μ, F_D is a stabilized decentralized static output feedback gain. Stop.

Step 5: If $\left\| X - P_i \right\| < \delta$, a pre-determined tolerance, go to *Step 6*, else set $X = P_i$ and $i = i + 1$, then go to *Step 2*.

Step 6: The system may be not stabilizable via decentralized static output feedback gain. Stop.

Detection Filter Problem

The state-space model of a linear time-invariant dynamic system with q failure modes can be modeled by (Beard, 1971; Jones, 1973; Massoumnia, 1986; White and Speyer, 1987; Douglas, 1996 and 1999)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + \sum_{i=1}^{q} F_i m_i(t) \\ y(t) &= Cx(t) \end{aligned} \tag{8}$$

where $x \in \mathcal{X}$, u $\in \mathcal{U}$ and $y \in \mathcal{Y}$ with $n = \dim(\mathcal{X})$, $p = \dim(\mathcal{U})$ and $m = \dim(\mathcal{Y})$. A is an $n \times n$ system state transmission matrix, B_u is an $n \times p$ input influence matrix, u is an $q \times 1$ input force vector, and C is an $m \times n$ output influence matrix. F_i is an $n \times 1$ fault direction vector, i=1,2,…,q, q is the number of fault directions and $m_i(t)$ is the ith arbitrary scalar function of time. When no faults occur, $m_i(t) = 0$. The fault directions F_i can be used to model actuator, sensor and component faults. A detailed treatment of all three failures can be found in Beard (1971) and Jones (1973). Consider the following full-order observer

$$\begin{aligned} \dot{\hat{x}}(t) &= Ax(t) + B_u u(t) - L(y(t) - Cx(t)) \\ r(t) &= C\hat{x}(t) - y(t) \end{aligned} \tag{9}$$

where $L \in \mathbf{R}^{n \times m}$ is the observer gain matrix, $r(t)$ is the residual outputs.

The state estimation error $\varepsilon(t) = \hat{x}(t) - x(t)$ dynamics are

$$\dot{\varepsilon}(t) = \left(A + LC\right)\varepsilon(t) - \sum_{i=1}^{q} F_i m_i(t) \tag{10}$$

$$r(t) = C\varepsilon(t) \tag{11}$$

If (C, A) is observable and L is chosen so that (A + LC) is stable, then in steady-state and in the absence of disturbances and modeling errors, the residual r is nonzero when a fault has occurred. But it is not enough. We also want to know which fault has occurred and this is what a detection filter is designed to do. In the papers by Massoumnia (1986) and Douglas (1999), the detection filter problem is interpreted in a geometric approach, where a set of (C, A) invariant subspaces $\mathcal{W}_i$ are found first and then L is the end product of an observer design algorithm. When $m_i(t) \neq 0$, the residual $r(t)$ remains in the output subspace $C\mathcal{W}_i$. Furthermore, the output subspace $C\mathcal{W}_1$, $C\mathcal{W}_2$, …, $C\mathcal{W}_q$ are independent so that $r(t)$ has a unique representation $r(t) = z_1 + z_2 + \ldots + z_q$ with $z_i \in C\mathcal{W}_q$. The fault is identified by projecting $r(t)$ onto each of the output subspaces $C\mathcal{W}_i$. The following statement summarizes the above detection filter problem in the geometric language.

Given the LTI system (8), the detection problem is to find a set of n-dimensional subspaces $\mathcal{W}_i$, i=1, 2, …, q, such that the following conditions are satisfied

Subspace invariance:

$$(A+LC)\mathcal{W}_i \subseteq \mathcal{W}_i \tag{12}$$

Fault inclusion

$$F_i \subseteq \mathcal{W}_i \tag{13}$$

Output separability

$$C\mathcal{W}_i \cap \sum_{j \neq i}^{q} C\,\mathcal{W}_j = \varnothing \tag{14}$$

for some matrix L with dimension $n \times m$.

Douglas (1996 and 1999) showed that the subspaces $\mathcal{W}_i$ are usually chosen as a set of mutual detectable, minimal unobservability subspaces or detection spaces. These subspaces are dependent on system matrices A, C and F_i and satisfy conditions (12), (13) and (14) such that the spectrum of (A + LC) may be placed arbitrarily. Given a set of detection spaces, the detection filter gain L can then be characterized easily as follows.

Let $\mathcal{W}_1$, $\mathcal{W}_2$, …, $\mathcal{W}_q$ be a set of (C, A) invariant subspaces that solve the detection filter problem and let the W_i: $\mathcal{W}_i \rightarrow \mathcal{X}$ be the insertion map. Let P_i: $\mathcal{W}_i \rightarrow \mathcal{W}_i$ be any projection where Ker(P_i) = Ker($C\mathcal{W}_i$) and let $\hat{F}_i$ decompose P_i as $\hat{F}_i F_i^T = P_i$ and $\hat{F}_i^T F_i = I$. Let H_i: $\mathcal{Y} \rightarrow \mathcal{Y}$ be another projection where Im(H_i) = $C\mathcal{W}_i$ and let $\tilde{H}_i$ be the associated natural projection that satisfies $\tilde{H}_i C T_i \hat{F}_i = I$ and $C T_i \hat{F} \tilde{H}_i = H_i$. The kernel of H_i and $\tilde{H}_i$ are satisfy

$$\sum_{j \neq i} C\,\mathcal{W}_i \in \mathrm{Ker}(H_i) = \mathrm{Ker}(\tilde{H}_i) \tag{15}$$

Also define the projection

$$H_0 = \left(I - \sum_{i=1}^{q} H_i \right) \tag{16}$$

and the associated natural projection $\tilde{H}_0$. Then the detection filter gain L can be parameterized as follows

$$L = \sum_{i=1}^{q} \left(-AW_i \hat{F}_i + W_i \alpha_i \right) \tilde{H}_i + \beta \tilde{H}_0 \tag{17}$$

for some β: Im(H_0) $\rightarrow \mathcal{X}$ and α_i: $C\mathcal{W}_i \rightarrow \mathcal{W}_i$ where i = 1,2,…,q.

In Eq. (17), α_i (i = 1, 2, …, q) and β are free parameters with compatible dimensions. These parameters should be chosen so that (A + LC) is stable. This is the only requirement to ensure that the residual $r(t)$ will stay in output subspace $C\mathcal{W}_i$ when $m_i(t)$ is nonzero. Douglas (1993) substituted Eq. (17) into the error dynamic system (Eq. (10) and (11)), finding that it is equivalent to the problem of decentralized static output feedback controller, shown as follows.

$$\dot{\varepsilon}(t) = \hat{A}\varepsilon(t) - \sum_{i=1}^{q} F_i m_i(t) + W_1 u_1 + \cdots + W_q u_q + u_0 \tag{18}$$

$$\begin{cases} \tilde{z}_1(t) = \tilde{H}_1 C\varepsilon(t) \\ \vdots \\ \tilde{z}_q(t) = \tilde{H}_q C\varepsilon(t) \end{cases} \tag{19}$$

$$\begin{cases} y_1(t) = \tilde{H}_1 C\varepsilon(t) \\ \vdots \\ y_q(t) = \tilde{H}_q C\varepsilon(t) \\ y_0(t) = \tilde{H}_0 C\varepsilon(t) \end{cases} \tag{20}$$

$$\begin{cases} u_1(t) = K_1 y_1(t) \\ \vdots \\ u_q(t) = K_q y_q(t) \\ u_0(t) = K_0 y_0(t) \end{cases} \tag{21}$$

where

$$\hat{A} = A + \sum_{i=1}^{q} \left(-AW_i \hat{F}_i \right) \tilde{H}_i C \tag{22}$$

The $\tilde{z}_1$, …, $\tilde{z}_q$ are system outputs and corresponding to the detection filter natural failure indications. The y_1, …, y_q are system observations and the u_1, …, u_q are system controlled inputs. The K_1, …, K_q and K_0 are constant decentralized controller gains which correspond to the detection filter gain parameters α_1, …, α_q and β and determine the closed-loop properties of the detection filter.

The system (18)~(21) is a decentralized static output feedback control problem (shown in Fig. 1). The task is to find unknown controllers K_0, K_1, …, K_q such that the closed-loop system is stable. As long as the closed-loop system is stable, i.e. all eigenvalues of (A + LC) are in the left complex plane, the effect of initial condition on the state estimation error $\varepsilon(t)$ (see Eq. (10)) will approach zero when time approaches infinity. But, if the poles of the closed-loop system are very close to the imaginary axle, it will take a long time to damp the effect of initial condition. As we know, the system output $z_i(t)$ will be nonzero when the ith fault occurs, i.e. $m_i(t) \neq 0$. It had better that $z_i(t)$ is mainly caused by fault, otherwise fault alarm may happen. Therefore, we hope that the closed-loop poles stay in the left of some negative number which is not very small. The decentralized controllers K_0, K_1, …, K_q can now be solved by iterative LMI approach introduced in Section 2.1.

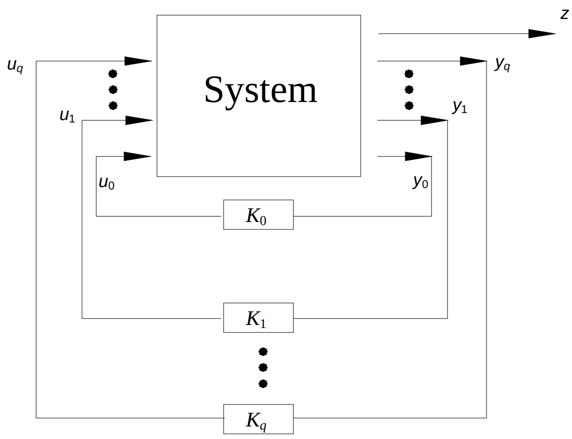

Fig. 1: Decentralized Control Diagram

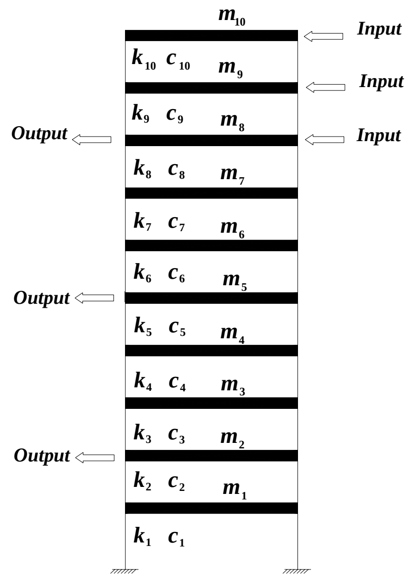

Fig. 2: 10-story shear type building

SIMULATION EXAMPLE

Shear walls are widely used in high-rise buildings to resist horizontal forces, especially in earthquake-prone region. Sometimes, the high-rise building with strong shear walls can be simply modeled as the shear type building. As shown in Fig. 2, it is a ten-story shear type building with 3 inputs and 3 outputs. Three inputs locate at the 8th, 9th and 10th story, and three outputs at the 2nd, 5th and 8th story. Measured outputs are displacement. Stiffness and mass of each story is 1000 N/m and 1 kg, respectively. Proportional Rayleigh damping is considered, i.e., $C = 0.1M + 0.01K$, where M, C, K are system mass, damping and stiffness matrix, respectively.

Suppose that after a moderate earthquake cracks only happen on the surface of shear walls in the first, fifth and eighth stories. Thus, in the simulation program, the stiffness of the first story is reduced 50% between 10s~40s, so is the fifth story between 30s~50s and the eighth story between 40s~60s. 5% rms noise is included in all measurements. The closed-loop poles are required to remain in the left of -0.3. After obtaining W_i, $\hat{F}_i$, $\tilde{H}_i$ (i = 1,2,3) and substituting them into iterative LMI algorithm, we have the following decentralized controller

$$F_D = \begin{bmatrix} -131.154 & 0 & 0 \\ -372.016 & 0 & 0 \\ -315.400 & 0 & 0 \\ 0 & -0.634 & 0 \\ 0 & -0.942 & 0 \\ 0 & 0 & -0.5145 \\ 0 & 0 & -0.8510 \end{bmatrix} \tag{23}$$

The most left pole of the closed-loop system is -0.425 ± j0.577, which is in the left of -0.3. The corresponding residual outputs are shown in Fig. 3, 4 and 5. Clearly, the story damages are identified correctly. Thus it can be concluded that the proposed method can be applied to structural damage detection.

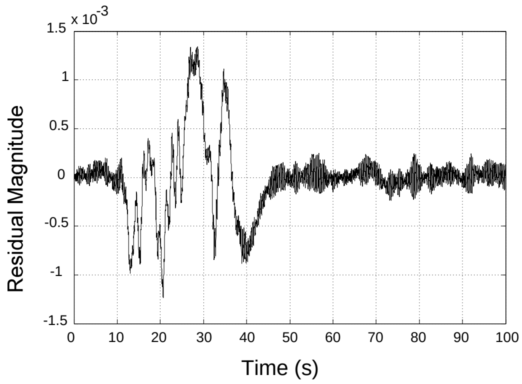

Fig. 3: The residual output for Story 1

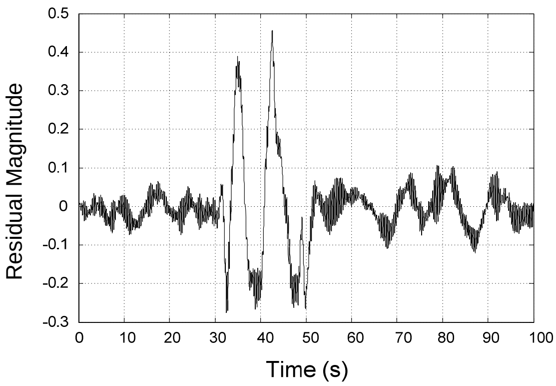

Fig. 4: The residual output for Story 5

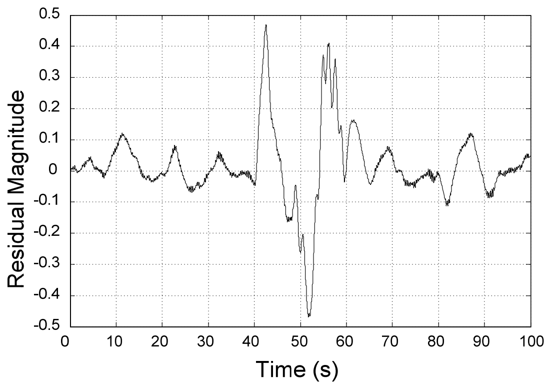

Fig. 5: The residual output for Story 8

CONCLUSIONS

In this paper, the observer-based fault detection and isolation problem is studied using detection filter concept and iterative LMI approach. The detection filter can not only detect the occurrence of structural damages, but also tell which one has damaged. The geometric interpretation of detection filter discloses the characteristics of observer gain L. The problem of finding detection filter gain is equivalent to that of finding a decentralized static output feedback controller gain which is a non-linear problem. In this paper, iterative LMI approach is used to find the decentralized controller and apply it to structural damage detection and isolation. The simulation example shows that the algorithm present in this paper can realize FDI goal.

ACKNOWLEDGMENTS

The authors wish to acknowledge the support of the *Texas Institute for the Intelligent Bio-Nano Materials and Structure for Aerospace Vehicles*, funded by NASA Cooperative Agreement no. NCC-1-02038. Partial travel grant to the second author by NSF for attending the US-Taiwan workshop is gratefully acknowledged.

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