A CONTINUOUS HYPERPLASTICITY MODEL FOR CLAYS UNDER CYCLIC LOADING

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ABSTRACT

The fact that soils can only exhibit truly elastic behaviour at very small strains is commonly known, so that soil behaviour under cyclic loading at small to moderate strains involves hysteretic behaviour. As the amplitude of cycling increases the soil stiffness decreases and the damping ratio increases. These are well established experimentally, but theories that successfully describe this behaviour are less well developed. This study presents a simple model for the behaviour of clay under cyclic loading which can capture the main features of small-strain cycling. An essential part of the model is that an effect of immediate stress history can be modelled. The model is described using the “continuous hyperplasticity” framework. Essentially this involves an infinite number of yield surfaces, thus allowing smooth transitions between elasticity and plasticity. The framework allows soil models to be developed in a relatively succinct mathematical form, since the entire constitutive behaviour can be determined through the specification of two scalar potentials. An implementation of the continuous hyperplasticity model is also carried out. Finally, comparisons between theoretical prediction and cyclic undrained triaxial compression test data of Bangkok Clay are presented.

Keywords: Hyperplasticity, Constitutive modelling, Cyclic triaxial test, Shear modulus, Damping ratio

INTRODUCTION

Plasticity theory is well-known that it is the important framework to explain the stress-strain behaviour of soils under monotonic loading. Many soil models based on plasticity have been developed since its first implementation to constitutive modelling of soils in the 1950’s. However, it has been less successful in describing soil behaviour under cyclic (unload-reloaded) loading. The principal problem is that, as is now well known empirically, soils exhibit elastic behaviour at only the very small strain and as the amplitude of strain cycling is increased, the secant stiffness steadily reduces and the damping increases. This pattern of behaviour does not match simply with plasticity theory, in which a finite elastic region is a fundamental part of the theory. Plasticity has been modified in a variety of ways to cope with this problem, with the two main approaches namely multi-surface plasticity and bounding surface plasticity. Of these multi-surface plasticity has more justification, since bounding surface plasticity cannot describe the well-established effects of immediate stress history. Multi-surface plasticity can, however, be rather cumbersome. An alternative is the “continuous hyperplasticity” approach, Puzrin and Houlsby (2001). This may be thought of as a variant of the multi-surface approach, in which the process is taken to its logical conclusion and an infinite number

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of surfaces are used. Continuous variations of stiffness and damping can be modelled. An advantage of the continuous hyperplasticity approach is that it is relatively compact mathematically. The entire constitutive response is specified through just two scalar functionals, thus avoiding the plethora of ad hoc assumptions that are often encountered in complex soil models.

The purpose here is to develop a simple model for soil behaviour especially clay under cyclic loading. Likitlersuang and Houlsby (2006) developed a series of hyperplasticity models for soil mechanics. The models based on multi-surface plasticity are selected for this study. In this paper we limit our attention to triaxial stress states, and so we do not address the shape of the yield surface in the octahedral plane.

**MODEL DESCRIPTION**

The model described here is an extension of a previous single surface model within the continuous hyperplasticity framework. This approach employs an infinite number of yield surfaces, which are expressed in terms of an internal coordinate $\eta$ (see Puzrin and Houlsby, 2001). In practice, however, the infinite number of surfaces have to be replaced by a finite number $N$ of surfaces. We label each surface $n$ ($1 \leq n \leq N$), and the factor $n/N$ plays the same role as $\eta$. In the following we present the model directly in terms of the finite number of surfaces, as this requires less sophisticated mathematics and leads more directly to the implementation. It should be borne in mind, however, that the underlying model involves an infinite number of surfaces, and this can be obtained by replacing $n/N$ by $\eta$, and by replacing summations by integrals. The model is formulated in terms of triaxial stress and strain variables:

$$p' = \frac{\sigma_1' + 2\sigma_3'}{3}, \quad q = \sigma_1 - \sigma_3$$

$$\varepsilon_p = \varepsilon_1 + 2\varepsilon_3, \quad \varepsilon_q = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$$

Volumetric and deviatoric plastic strains related to the $n^{th}$ plastic mechanism are indicated as $\alpha_p^{(n)}$ and $\alpha_q^{(n)}$ respectively. The specification of two scalar functions, a Gibbs free energy function $g$ or, alternatively, a Helmholtz free energy function $f$

$$g = g\left(p, q, \alpha_p^{(1)}, \alpha_p^{(2)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \alpha_q^{(2)}, \ldots, \alpha_q^{(n)}\right)$$

$$f = f\left(\varepsilon_p, \varepsilon_q, \alpha_p^{(1)}, \alpha_p^{(2)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \alpha_q^{(2)}, \ldots, \alpha_q^{(n)}\right)$$

and dissipation functions $d$ or yield functions $y$ based on Gibbs free energy and Helmholtz free energy

$$d = d^g\left(p, q, \alpha_p^{(1)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \ldots, \alpha_q^{(n)}, \dot{\alpha}_p^{(1)}, \ldots, \dot{\alpha}_p^{(n)}, \dot{\alpha}_q^{(1)}, \ldots, \dot{\alpha}_q^{(n)}\right) \geq 0$$

$$d = d^f\left(\varepsilon_p, \varepsilon_q, \alpha_p^{(1)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \ldots, \alpha_q^{(n)}, \dot{\alpha}_p^{(1)}, \ldots, \dot{\alpha}_p^{(n)}, \dot{\alpha}_q^{(1)}, \ldots, \dot{\alpha}_q^{(n)}\right) \geq 0$$

$$y = y^g\left(p, q, \alpha_p^{(1)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \ldots, \alpha_q^{(n)}, \chi_p^{(1)}, \ldots, \chi_p^{(n)}, \chi_q^{(1)}, \ldots, \chi_q^{(n)}\right) = 0$$

$$y = y^f\left(\varepsilon_p, \varepsilon_q, \alpha_p^{(1)}, \ldots, \alpha_p^{(n)}, \alpha_q^{(1)}, \ldots, \alpha_q^{(n)}, \chi_p^{(1)}, \ldots, \chi_p^{(n)}, \chi_q^{(1)}, \ldots, \chi_q^{(n)}\right) = 0$$
are sufficient to define completely the constitutive behaviour. The following two models are used here:

**Gibbs free energy function**

1. Linear volumetric stress-strain relationship

\[
g = -\frac{p^2}{2K} - \frac{q^2}{6G} - \left(p\alpha_p + q\alpha_q\right) + \frac{1}{N} \sum_{i=1}^{n} \left(\frac{1}{2} H_p^{(i)}\alpha_p^{(i)} + \frac{1}{2} H_q^{(i)}\alpha_q^{(i)}\right)
\]  

(6)

where \(H_p^{(i)}, H_q^{(i)}\) are the hardening modulus related to the \(i^{th}\) mechanism and \(K, G\) are bulk and shear modulus respectively.

2. Logarithmic volumetric stress-strain relationship

\[
g = -kp_0\log\left(\frac{p}{p_0}\right) - \frac{q^2}{6G} - \left(p\alpha_p + q\alpha_q\right) + \frac{1}{N} \sum_{i=1}^{n} \lambda_p^{(i)} p_0 \exp\left(\frac{\alpha_p^{(i)} + 3h_q^{(i)}\alpha_q^{(i)}^2/2}{\lambda_q^{(i)}}\right)
\]  

(7)

where \(\lambda(x) = x \log(x) - x\), \(\lambda^{(i)}\) is nonlinear hardening function, \(p_0\) is reference pressure.

**Dissipation function or yield function**

\[
d = \frac{1}{N} \sum_{i=1}^{n} c^{(i)} \sqrt{\dot{\alpha}_p^{(i)}^2 + M^2 \dot{\alpha}_q^{(i)}^2} = 0
\]  

(8a)

\[
y^{(i)} = \sqrt{\lambda_p^{(i)} + \lambda_q^{(i)} / M^2} - c^{(i)} = 0
\]  

(8b)

where \(M\) is the value that the stress ratio \(q/p'\) attains at critical state conditions and \(c^{(i)}\) represents the range of each yield surface. The normalisation term \(1/N\) in (6), (7) and (8) makes the formulation independent on the number of surfaces. The definitions of constrains enable the plastic strains (a summation of all internal variables):

\[
c_p^* = \alpha_p - \sum_{i=1}^{n} \alpha_p^{(i)} = 0
\]  

(9a)

\[
c_q^* = \alpha_q - \sum_{i=1}^{n} \alpha_q^{(i)} = 0
\]  

(9b)

For the linear model, the yield surfaces exhibit kinematic hardening which is given by the term \(\frac{1}{2} H_p^{(i)}\alpha_p^{(i)}^2\) and \(\frac{1}{2} H_q^{(i)}\alpha_q^{(i)}^2\) for consolidation and shear mode respectively. The expressions for the variation of the consolidation hardening modulus are:
\[ H_p^{(i)} = \frac{K \left(1 - \frac{i}{N}\right)^{b_p}}{2(a_p - 1)} \]  

where \( K \) is the initial (elastic) bulk modulus which is related to the slope of the swelling line, \( \kappa \) by \( K = p'(1+e)/\kappa' \), and \( a_p, b_p \) are nonlinear hardening parameters and parameter-\( r \) control the plastic bulk modulus which is related to the slope of the normally consolidated line, \( \lambda \). The shear hardening function is simply expressed to be hyperbolic function which is

\[ H_q^{(i)} = \frac{3G \left(1 - \frac{i}{N}\right)^{b_q}}{2(a_q - 1)} \]

where \( G \) is the initial (elastic) shear modulus and \( a_q, b_q \) are hyperbolic hardening parameters. However, the shear modulus \( G \) can be assumed to be linearly proportional to the mean stress that is \( G = g_x p \), where \( g_x \) is gradient of shear modulus.

For the logarithmic stress-strain model, the kinematic hardening is defined by term

\[ \lambda^{(i)} p_0 \exp \left( \frac{\alpha_p^{(i)} + 3h^{(i)} \alpha_q^{(i)} / 2}{\lambda^{(i)}} \right) \]  

A possible function \( \lambda^{(i)} \) is assumed in the form of the power function:

\[ \lambda^{(i)} = (\lambda - \kappa)(n + 1) \left(\frac{i}{N}\right)^n \]

There are three parameters (\( \kappa, \lambda, \) and \( n \)) for the logarithmic hardening function. The \( \kappa \) and \( \lambda \) are the conventional soil parameters, which respectively represent the slope of swelling line and normally consolidated line. The \( n \) value can be simply obtained using the measure of the slope at half of the maximum past pressure (see Likitlersuang and Houlsby, 2006). The shearing behaviour is still assumed that the shear modulus is linearly proportional to pressure \( (G = g_x p) \) and the shear hardening function also uses the form \( h^{(i)} = \frac{g_x \left(1 - \frac{i}{N}\right)^b}{2(a_q - 1)} \).

To introduce the difference between compression and extension, the critical stress ratio \( M \) is given by:

\[ M = \frac{1}{2} \left[ (M_c + M_e) + (M_c - M_e) \text{sgn}(\dot{\alpha}_q^{(n)}) \right] \]

**MODEL IMPLEMENTATION**

The linear stress-strain model in (6) and (8) was first implemented by Likitlersuang and Houlsby (2004). The model, which is called the kinematic hardening modified Cam-clay (KHMCC) model,
was applied to the numerical calculation with the rate-dependent algorithm. The monotonic loading predictions on Bangkok Clay were carried out using the parameter values given in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Physical meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>5200 (kPa)</td>
<td>Initial (elastic) bulk modulus</td>
</tr>
<tr>
<td>$a_p$</td>
<td>2.0</td>
<td>Nonlinear kinematic hardening parameter for consolidation behaviour</td>
</tr>
<tr>
<td>$b_p$</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>$g_x$</td>
<td>60</td>
<td>Elastic shear modulus gradient</td>
</tr>
<tr>
<td>$a_q$</td>
<td>3.5</td>
<td>Nonlinear kinematic hardening parameter for shear behaviour</td>
</tr>
<tr>
<td>$b_q$</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>0.9</td>
<td>Slope of critical state line in $q - p'$ plane</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-</td>
<td>Viscosity coefficient (for rate-dependent)</td>
</tr>
</tbody>
</table>

Although the overall prediction of undrained and drained triaxial tests are good, there are still some shortcomings of the KHMCC model such as the softening behaviour of heavily overconsolidated clay and anisotropic behaviour. These are needed to consider for the further work.

MODEL PREDICTION FOR CYCLIC UNDRAINED TRIAXIAL TESTS OF BANGKOK CLAY

The most important advantage of the multi-surface plasticity model is that it can simulate the smooth transition of stiffness before failure occurs. This advantage is crucial in construction work, because the stress-strain behaviour is actually limited under a certain value (i.e. the strength divided by a safety factor), which is much lower than the peak strength. In recent years, models for the small strain behaviour of soils have been developed independently of the model of prediction of strength behaviour, as there is no theoretical framework to describe both behaviours in an uncomplicated way. However, hyperplasticity with a continuous kinematic hardening function can explain both.

The data on cyclic loading from Bangkok clay can be used to compare with the model prediction. The testing programme concerns the undrained cyclic behaviour (Teachavorasinsku et al., 2001). Table 2 presents the definitions of the cyclic stress-strain parameters used in this research. For the simulation of cyclic tests, the shear modulus gradient ($g_x$) is taken at a slightly higher value than for static tests, i.e. $g_x = 150$ in cyclic tests, while $g_x = 60$ in static tests. For other soil parameters, they are defined as in Table 1.

Figures 2 and 3 present the simulated result of cyclic undrained shear tests with varying confining pressure ($p_c$) of 50kPa and 100kPa respectively. The simulated stress paths are presented in Figure 2(a) and 3(a) for the case of 50kPa and 100kPa confinement respectively. The shear stress-strain curves for 50kPa and 100kPa confinement are also shown in Figure 2(b) and 3(b) respectively. Due to the lack of stress-strain data from the original research (Teachavorasinsku et al., 2001), the comparison between model prediction and test results can not be made.
\[ \varepsilon_3 = -\varepsilon_1 \]

\[ \varepsilon_1 \]

\[ \gamma_{SA} \]

\[ A_L \]

\[ A_E \]

\[ h = \frac{A_L}{4\pi A_E} \]

Figure 1 (a) Mohr’s circle represents the strain parameters for undrained condition; (b) Definition of damping ratio

Table 2 The definitions of cyclic stress-strain parameters

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \gamma_{SA} ]</td>
<td>Single amplitude shear strain</td>
</tr>
<tr>
<td>[ \frac{\gamma_{SA}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} ] in undrained condition [ \gamma_{SA} = 1.5\varepsilon_s ]</td>
<td></td>
</tr>
<tr>
<td>[ G = \frac{E}{2(1-v)} ] in undrained condition [ G = \frac{q}{\varepsilon_s} ]</td>
<td>Secant shear modulus</td>
</tr>
<tr>
<td>[ h = \frac{A_L}{4\pi A_E} ] where [ A_L ] and [ A_E ] represent the area of hysteresis loop and the area of elastic zone (show in Figure 1)</td>
<td>Damping ratio</td>
</tr>
</tbody>
</table>

However, the comparisons on the variation of shear modulus and damping ratio with single amplitude shear strain can be carried out. The prediction of cyclic undrained triaxial tests are good for both the normally consolidated sample \( (p_c = 100\text{kPa}) \) and lightly overconsolidated sample \( (p_c = 50\text{kPa}) \) as presented in Figure 4 and 5 respectively. The \( N \)-value in Figures 4(a) and 5(a) represents the number of testing cycles. Although, the simulations has not been theoretically processed until failure (excess pore pressure develops due to the cyclic load), the first ten cycles are sufficient to represent the cyclic behaviour. The effect of viscous behaviour on cyclic properties can actually be investigated by the KHMCC model as well. However, because of the inconsiderable amount of the frequency difference from the original research, it does not show a significant effect.

The prediction of the cyclic test at small strains is the most important advantage of the multi-surface plasticity model. In particular, the rate-dependent model shows the deformed elliptical shape of hysteresis loops during the cyclic load. The area of the hysteresis loop becomes larger when the number of cycles increase. This evidence is exhibited on the damping ratio Figures 4(b) and 5(b).
Figure 2. Simulation of the KHMCC model for cyclic undrained shear test varying shear stress magnitude \((q/p)\) at confining pressure \((p_c) = 50\text{kPa}\);
(a) cyclic stress paths, (b) cyclic shear stress-strain curves

Figure 3. Simulation of the KHMCC model for cyclic undrained shear test varying shear stress magnitude \((q/p)\) at confining pressure \((p_c) = 100\text{kPa}\);
(a) cyclic stress paths, (b) cyclic shear stress-strain curves
Figure 4. Prediction of cyclic undrained shear test varying shear stress magnitude ($q/p$) at confining pressure ($p_c$) = 50kPa, $N$ = number of cycles;
(a) Plot of shear modulus ($G$) vs. single amplitude shear strain ($\gamma$)
(b) Plot of damping ratio ($h$) vs. single amplitude shear strain ($\gamma$)
Figure 5. Prediction of cyclic undrained shear test varying shear stress magnitude \((q/p)\) at confining pressure \((p_c) = 100\text{kPa}\). \(N\) = number of cycles;
(a) Plot of shear modulus \((G)\) vs. single amplitude shear strain \((\gamma)\)
(b) Plot of damping ratio \((h)\) vs. single amplitude shear strain \((\gamma)\)
CONCLUSION

A model for the cyclic behaviour of clay under triaxial conditions has been presented. The model successfully describes typical trends of cyclic behaviour especially for undrained test, including typical variation of shear modulus and damping ratio for different strain amplitude as well as different confinement. The comparisons with the experimental results are carried out on Bangkok Clay.

REFERENCES:


